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Dipolarizations in quadratic Lie algebras and homogeneous parakähler manifolds

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Abstract
A Lie algebra $\mathfrak{g}$ with a nondegenerate symmetric and invariant bilinear form is called quadratic. A dipolarization in a Lie algebra $\mathfrak{g}$ is the two polarizations $\mathfrak{g}^{\pm}$ in $\mathfrak{g}$ at a common linear form on $\mathfrak{g}$ satisfying $\mathfrak{g} = \mathfrak{g}^{+} + \mathfrak{g}^{-}$. In this paper, we study dipolarizations in extended Heisenberg algebras, a subclass of quadratic Lie algebras, and homogeneous parakähler manifolds associated with these dipolarizations.

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Introduction

A Lie algebra $\mathfrak{g}$ with a nondegenerate symmetric and invariant bilinear form B is called a *quadratic Lie algebra* and B is called the *invariant scalar product* on $\mathfrak{g}$. It is well known that quadratic Lie algebras play a privileged role in physics. For example, Nappi and Witten showed in [1] that a non-semisimple quadratic Lie algebra can be allowed for a Sugawara-type construction. The construction in [1] was quickly generalized by Sfetsos [2] to Abelian extensions of m-dimensional Euclidean algebras. In [3], Mohammedi spelled out that asking for a Sugawara construction is equivalent to demanding that the Lie algebra possesses an invariant scalar product. So it seems natural to investigate further the properties of quadratic Lie algebras.

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$ (=$\mathbb{R}$ or $\mathbb{C}$), $\mathfrak{g}^{\pm}$ be two subalgebras of $\mathfrak{g}$ and f a linear form on $\mathfrak{g}$. Kaneyuki [4], defines a *dipolarization* in $\mathfrak{g}$ as a triple $\{\mathfrak{g}^{\pm}, f\}$ which satisfies some conditions (cf definition 1.7). A dipolarization is called *symmetric* if the two subalgebras $\mathfrak{g}^{\pm}$ are isomorphic to each other as Lie algebras. Otherwise it is called *nonsymmetric*. A dipolarization $\{\mathfrak{g}^{\pm}, f\}$ in Lie algebra $\mathfrak{g}$ is called *trivial* if $\mathfrak{g}^{+} = \mathfrak{g}^{-} = \mathfrak{g}$ and $f = 0$.

Let $\mathfrak{g}$ be a real Lie algebra and ρ an alternating 2-form on $\mathfrak{g}$. A weak dipolarization, more generally, was introduced by Kaneyuki [4]. A triple $\{\mathfrak{g}^{\pm}, \rho\}$ is called a *weak dipolarization* in $\mathfrak{g}$, if (1) $\mathfrak{g} = \mathfrak{g}^+ + \mathfrak{g}^-$, (2) $\rho(x, \mathfrak{g}) = 0$ if and only if $x \in \mathfrak{g}^+ \cap \mathfrak{g}^-$, (3) $\rho(\mathfrak{g}^+, \mathfrak{g}^+) = \rho(\mathfrak{g}^-, \mathfrak{g}^-) = 0$ and (4) $\rho([x, y], z) + \rho([y, z], x) + \rho([z, x], y) = 0, \forall x, y, z \in \mathfrak{g}$. Clearly a dipolarization $\{\mathfrak{g}^{\pm}, f\}$ is a weak dipolarization just by taking $\mathrm{d}f$ as ρ. If $\mathfrak{g}$ is semisimple, furthermore, weak dipolarizations reduce to dipolarizations. A homogeneous parakähler structure is perfectly described by a weak dipolarization (see [4]). A *homogeneous parakähler manifold* is, by definition, a homogeneous symplectic manifold (of a Lie group G) which admits a pair of transversal Lagrangian foliations (see [4]).

The notion of dipolarizations in Lie algebras is also closely related to that of polarizations (see definition 1.6), which plays an important role in the theory of unitary representations of Lie groups. Lemma 1.9 shows us a method to construct polarizations in Lie algebras.

In [4], Kaneyuki obtained a remarkable class of symmetric dipolarizations in real semisimple Lie algebras by using gradations. Hou *et al* [5] gave the inductive classification of polarizations on semisimple Lie algebras. So they [5] settled, in some sense, the classification problem on homogeneous parakähler manifolds. The first example of nonsymmetric dipolarization was given in [6]. Furthermore, [7] gave a large class of nonsymmetric dipolarizations in solvable complete Lie algebras. In [8], we showed that there exist dipolarizations in quadratic Lie algebras whose Cartan subalgebras consist of semisimple elements and gave some general results on the classification of dipolarizations in quadratic Lie algebras. We also determined all of the dipolarizations in four-dimensional extended Heisenberg algebra and considered the homogeneous parakähler manifolds associated with these dipolarizations.

In this paper, we construct six classes of dipolarizations in $(2n+2)$-dimensional extended Heisenberg algebra and study the homogeneous parakähler manifolds associated with these dipolarizations. We find two facts which are different from the case in semisimple Lie algebras. The first one is that there exist nilpotent characteristic elements corresponding to some symmetric or nonsymmetric dipolarizations in extended Heisenberg algebras. The other one is that there exist symmetric and nonsymmetric dipolarizations in extended Heisenberg algebras at the same time.

1. The definitions and some fundamental results

Definition 1.1 (cf [9])**.** *Let M be a smooth symplectic manifold, if M admits two smooth transversal Lagrangian foliations, then M is called a* parakähler manifold.

Definition 1.2 (cf [10])**.** *A parakähler manifold M is called* homogeneous *if the action of $G(M)$ on M is transitive, where $G(M)$ is a (finite-dimensional) Lie group which consists of all the diffeomorphisms of M preserving both the symplectic structure and the two foliations.*

To describe homogeneous manifolds, it is natural to consider the algebraic conditions for the existence of such a structure. In 1990, Kaneyuki considered this problem for homogeneous parakähler manifolds. For this he introduced several new notions in Lie algebras.

Definition 1.3 (cf [4])**.** *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$, a triple $\{\mathfrak{g}^+, \mathfrak{g}^-, \rho\}$ is called a* weak dipolarization *in $\mathfrak{g}$, where $\mathfrak{g}^{\pm}$ are two subalgebras of $\mathfrak{g}$, ρ is an alternative 2-form on $\mathfrak{g}$, and the following conditions are satisfied:*

(1) $\mathfrak{g} = \mathfrak{g}^+ + \mathfrak{g}^-$,
(2) $\rho(x, \mathfrak{g}) = 0$ *if and only if* $x \in \mathfrak{g}^+ \cap \mathfrak{g}^-$,

(3) $\rho(\mathfrak{g}^+, \mathfrak{g}^+) = \rho(\mathfrak{g}^-, \mathfrak{g}^-) = 0$,
(4) $\rho([x, y], z) + \rho([y, z], x) + \rho([z, x], y) = 0, \qquad \forall x, y, z \in \mathfrak{g}$.

Lemma 1.4 ([11])**.** *Let* $\mathfrak{g}$ *be a Lie algebra over a field* $\mathbb{F}$ *and* $\{\mathfrak{g}^+, \mathfrak{g}^-, \rho\}$ *be a weak dipolarization in* $\mathfrak{g}$. *Then we have* $\dim\mathfrak{g}^+ = \dim\mathfrak{g}^-$.

The following theorem is the main result of [4].

Theorem 1.5 ([4])**.** *Let* G *be a (real) Lie group,* H *be a closed subgroup of* G. *Then there exists a* G*-invariant parakähler structure on the coset* G/H *if and only if there exists a weak dipolarization* $\{\mathfrak{g}^+, \mathfrak{g}^-, \rho\}$ *in the Lie algebra* $\mathfrak{g} = \mathrm{Lie}\, G$, *such that* $\mathfrak{g}^+ \cap \mathfrak{g}^- = \mathfrak{h} = \mathrm{Lie}\, H$, *and the following two conditions are satisfied:*

$$(*) \quad \mathrm{Ad}(h)\mathfrak{g}^{\pm} = \mathfrak{g}^{\pm}, \qquad \forall h \in H,$$
$$(**) \quad \rho(\mathrm{Ad}(h)x, \mathrm{Ad}(h)y) = \rho(x, y), \qquad \forall x, y \in \mathfrak{g}, h \in H.$$

Remark. If H is connected then conditions $(*)$ and $(**)$ are not necessary.

Kaneyuki introduced another notion—dipolarizations in Lie algebras, which is simpler but enough for many cases.

Definition 1.6 (cf [12])**.** *Let* $\mathfrak{g}$ *be a Lie algebra over a field* $\mathbb{F}$ *and* $f \in \mathfrak{g}^*$ *(dual of* $\mathfrak{g}$*). A subalgebra* $\mathfrak{p}$ *of* $\mathfrak{g}$ *is called a* polarization *in* $\mathfrak{g}$ *at* f, *if* $\mathfrak{p}$ *satisfies:*

(1) $f([\mathfrak{p}, \mathfrak{p}]) = 0$,
(2) $\mathfrak{p}$ *is a maximal subspace satisfying* (1). *That is, if* $\mathfrak{p}'$ *is another subspace of* $\mathfrak{g}$ *satisfying* $\mathfrak{p} \subset \mathfrak{p}'$ *and* $f([\mathfrak{p}', \mathfrak{p}']) = 0$, *then* $\mathfrak{p} = \mathfrak{p}'$.

In general, we will denote polarization by a pair $\{\mathfrak{p}, f\}$.

Definition 1.7 (cf [4])**.** *Let* $\mathfrak{g}$ *be a Lie algebra over a field* $\mathbb{F}$. *A triple* $\{\mathfrak{g}^+, \mathfrak{g}^-, f\}$ *is called a* dipolarization *(over* $\mathbb{F}$*) in* $\mathfrak{g}$, *if the following conditions are satisfied:*

(1) $\mathfrak{g}^+$ *and* $\mathfrak{g}^-$ *are two subalgebras of* $\mathfrak{g}$ *and* f *is an* $\mathbb{F}$*-linear form on* $\mathfrak{g}$.
(2) Let $\mathfrak{h} = \mathfrak{g}^+ \cap \mathfrak{g}^-$, *then* $f([X, \mathfrak{g}]) = 0$ *if and only if* $X \in \mathfrak{h}$,
(3) $f([\mathfrak{g}^+, \mathfrak{g}^+]) = f([\mathfrak{g}^-, \mathfrak{g}^-]) = 0$,
(4) $\mathfrak{g} = \mathfrak{g}^+ + \mathfrak{g}^-$.

Remark. Let $\{\mathfrak{g}^+, \mathfrak{g}^-, f\}$ be a dipolarization in $\mathfrak{g}$. Define an alternative 2-form by $\rho(x, y) = f([x, y])$. Then it is easily seen that $\{\mathfrak{g}^+, \mathfrak{g}^-, \rho\}$ is a weak dipolarization in $\mathfrak{g}$. On the other hand, if $\mathfrak{g}$ is a semisimple Lie algebra over $\mathbb{C}$ or $\mathbb{R}$, then any weak dipolarization in $\mathfrak{g}$ can be induced from a dipolarization in the manner given above. This is because the Killing form is nondegenerate.

Combining lemma 1.4 with the above remark, we have

Corollary 1.8 ([11])**.** *Let* $\{\mathfrak{g}^+, \mathfrak{g}^-, f\}$ *be a dipolarization in a Lie algebra* $\mathfrak{g}$. *Then* $\dim\mathfrak{g}^+ = \dim\mathfrak{g}^-$.

A dipolarization $\{\mathfrak{g}^+, \mathfrak{g}^-, f\}$ is called *symmetric* if $\mathfrak{g}^+$ is isomorphic to $\mathfrak{g}^-$ as a Lie algebra. Otherwise it is called *nonsymmetric*. It is called *trivial* if $\mathfrak{g}^+ = \mathfrak{g}^- = \mathfrak{g}$ and $f = 0$.

The relation of polarizations and dipolarizations in Lie algebras is as the following:

Lemma 1.9 ([5])**.** *Let* $\mathfrak{g}$ *be a Lie algebra over a field* $\mathbb{F}$. *Then* $\{\mathfrak{g}^+, \mathfrak{g}^-, f\}$ *is a dipolarization in* $\mathfrak{g}$ *if and only if*

(1) $\{\mathfrak{g}^+, f\}$ *and* $\{\mathfrak{g}^-, f\}$ *are two polarizations in* $\mathfrak{g}$, *and*
(2) $\mathfrak{g} = \mathfrak{g}^+ + \mathfrak{g}^-$.

Definition 1.10. *A Lie algebra* $\mathfrak{g}$ *with a nondegenerate symmetric and invariant bilinear form* B *is called a* quadratic Lie algebra.

Now we generalize the concept of *characteristic element* (cf [10]) to the quadratic Lie algebras endowed with dipolarizations.

Definition 1.11. *Let* $(\mathfrak{g}, B)$ *be a quadratic Lie algebra with invariant scalar product* B *over* $\mathbb{F}$, *and let* $\{\mathfrak{g}^\pm, f\}$ *be a dipolarization in* $\mathfrak{g}$. *Then there exists a unique element* z *such that*

$$B(z, x) = f(x) \qquad x \in \mathfrak{g}.$$

Then we call z *the* characteristic element *of the dipolarization* $\{\mathfrak{g}^\pm, f\}$.

In view of the above definition, we often say a dipolarization $\{\mathfrak{g}^\pm, z\}$ instead of $\{\mathfrak{g}^\pm, f\}$. In the same way one can define the characteristic element of a polarization in $\mathfrak{g}$.

Lemma 1.12. *Let* $(\mathfrak{g}, B)$ *be a quadratic Lie algebra with invariant scalar product* B *over* $\mathbb{F}$, *and let* $f(x) = B(z, x)$ $(z \neq 0 \in \mathfrak{g})$ *for any* $x \in \mathfrak{g}$. *Then* $\mathfrak{g}^f = \{x \in \mathfrak{g} : f([x, \mathfrak{g}]) = 0\}$ *coincides with the centralizer* $\mathbf{C}_{\mathfrak{g}}(z)$ *of* z *in* $\mathfrak{g}$.

Proof. Noting the invariance and nondegeneracy of B, we have that

$$\begin{aligned} & x \in \mathfrak{g}^f \\ \Longleftrightarrow\ & f([x, \mathfrak{g}]) = B(z, [x, \mathfrak{g}]) = 0 \\ \Longleftrightarrow\ & B([z, x], \mathfrak{g}) = 0 \Longleftrightarrow [z, x] = 0 \Longleftrightarrow x \in \mathbf{C}_{\mathfrak{g}}(z). \end{aligned}$$

This concludes the assertion. □

2. Dipolarizations in extended Heisenberg algebras

Let

$$\mathfrak{g} = \mathbb{F}t \oplus \sum_{i=1}^{n} \mathbb{F}e_i \oplus \sum_{i=1}^{n} \mathbb{F}\varepsilon_i \oplus \mathbb{F}c \ (\mathbb{F} = \mathbb{C} \text{ or } \mathbb{R})$$

as vector spaces. Define Lie bracket on $\mathfrak{g}$ by setting

$$[t, e_i] = e_i \qquad [t, \varepsilon_i] = -\varepsilon_i \qquad [e_i, \varepsilon_j] = \delta_{ij}c \qquad (i, j = 1, 2, \ldots, n)$$

and bilinear form on $\mathfrak{g}$ by

$$B(t, c) = 1 \qquad B(e_i, \varepsilon_j) = \delta_{ij}.$$

Then $\mathfrak{g}$ is a solvable quadratic Lie algebra with a Cartan subalgebra $\mathfrak{h} = \mathbb{F}t + \mathbb{F}c$ which consists of semisimple elements. We call it *extended Heisenberg algebra*.

In [8], we showed that there exist dipolarizations in quadratic Lie algebras whose Cartan subalgebras consist of semisimple elements. Particularly, we determined all of the dipolarizations in four-dimensional extended Heisenberg algebra and studied the homogeneous parakähler manifolds associated with these dipolarizations. In this section, we construct six classes of nontrivial dipolarizations in general extended Heisenberg algebras.

Let $g_0 = \mathbb{F}\big(t + \sum_{i=1}^n \lambda_i e_i + \sum_{j=1}^n \mu_j \varepsilon_j\big)$, $g_1 = \sum_{i=1}^n \mathbb{F}e_i$, $g_2 = \sum_{i=1}^n \mathbb{F}\varepsilon_i$, $g_{1k} = \sum_{i\neq k} \mathbb{F}e_i$ $(1 \leqslant k \leqslant n)$, $g_{2k} = \sum_{i\neq k} \mathbb{F}\varepsilon_i$ $(1 \leqslant k \leqslant n)$, $g_{1S} = \sum_{i\in S} \mathbb{F}e_i$, $\overline{g}_{1S} = \sum_{i\in \overline{S}} \mathbb{F}e_i$,

Table 1. The dipolarizations in general extended Heisenberg algebras.

	g^+	g^-	$\mathfrak{U}$
(1)	$g_0 \oplus \mathbb{F}c \oplus g_1$	$g_0 \oplus \mathbb{F}c \oplus g_2$	$\mathfrak{U}_0$
(2)	$g_1 \oplus g_{2k} \oplus \mathbb{F}c \oplus \mathbb{F}(t + \mu\varepsilon_k)$	$g_1 \oplus g_{2k} \oplus \mathbb{F}c \oplus \mathbb{F}(t + \mu'\varepsilon_k)$	$\mathfrak{U}_{1k}$
(3)	$g_{1k} \oplus g_2 \oplus \mathbb{F}c \oplus \mathbb{F}(t + \mu e_k)$	$g_{1k} \oplus g_2 \oplus \mathbb{F}c \oplus \mathbb{F}(t + \mu' e_k)$	$\mathfrak{U}_{2k}$
(4)	$g_0 \oplus g_{1S} \oplus \overline{g}_{2S} \oplus \mathbb{F}c$	$g_0 \oplus \overline{g}_{1S} \oplus g_{2S} \oplus \mathbb{F}c$	$\mathfrak{U}_0$
(5)	$g_1 \oplus g_{2k} \oplus \mathbb{F}c \oplus \mathbb{F}(t + \mu\varepsilon_k)$	$g_1 \oplus g_2 \oplus \mathbb{F}c$	$\mathfrak{U}_{1k}$
(6)	$g_{1k} \oplus g_2 \oplus \mathbb{F}c \oplus \mathbb{F}(t + \mu e_k)$	$g_1 \oplus g_2 \oplus \mathbb{F}c$	$\mathfrak{U}_{2k}$

$g_{2S} = \sum_{i\in S} \mathbb{F}\varepsilon_i$ and $\overline{g}_{2S} = \sum_{i\in \overline{S}} \mathbb{F}\varepsilon_i$ be the subspaces of $\mathfrak{g}$, where S is a subset of $\{1, 2, \ldots, n\}$ and $\overline{S}$ the complementary to S in $\{1, 2, \ldots, n\}$, $\lambda_i, \mu_j \in \mathbb{R}$.

Let $\mathfrak{U}_0 = \left\{z \mid z = t_1 t + t_2 c + t_1 \sum_{i=1}^n \lambda_i e_i + t_1 \sum_{j=1}^n \mu_j \varepsilon_j, t_1 \neq 0\right\}$, $\mathfrak{U}_{1k} = \{z \mid z = t_1 e_k + t_2 c, t_1 \neq 0\}$ $(1 \leqslant k \leqslant n)$, $\mathfrak{U}_{2k} = \{z \mid z = t_1\varepsilon_k + t_2 c, t_1 \neq 0\}$ $(1 \leqslant k \leqslant n)$.

By a straightforward calculation, we get the above six classes of nontrivial dipolarizations in general extended Heisenberg algebras in table 1. In table 1, $\mathfrak{U}$ denotes the set of the characteristic elements of a polarization in $\mathfrak{g}$, $\mu, \mu' \in \mathbb{R}$ and $\mu \neq \mu'$.

We give the proof only to (1). Clearly $g^+ = g_0 \oplus \mathbb{F}c \oplus g_1$ and $g^- = g_0 \oplus \mathbb{F}c \oplus g_2$ are two subalgebras of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{g}^+ + \mathfrak{g}^-$,

$$\mathfrak{h} = \mathfrak{g}^+ \cap \mathfrak{g}^- = \mathbb{F}\left(t + \sum_{i=1}^n \lambda_i e_i + \sum_{j=1}^n \mu_j \varepsilon_j\right) \oplus \mathbb{F}c$$

$$[\mathfrak{g}^+, \mathfrak{g}^+] = \sum_{k=1}^n \mathbb{F}(e_k - \mu_k c) \qquad [\mathfrak{g}^-, \mathfrak{g}^-] = \sum_{k=1}^n \mathbb{F}(\varepsilon_k - \lambda_k c).$$

Let $f_z(x) = B(z, x)$ $(z \in \mathfrak{U}_0)$ for any $x \in g$. Then $f_z([\mathfrak{g}^+, \mathfrak{g}^+]) = f_z([\mathfrak{g}^-, \mathfrak{g}^-]) = 0$. Let $x = x_t t + x_c c + + \sum_{i=1}^n l_i e_i + \sum_{j=1}^n m_j \varepsilon_j$ be any element in $\mathfrak{g}$ and $z = t_1 t + t_2 c + t_1 \sum_{i=1}^n \lambda_i e_i + t_1 \sum_{j=1}^n \mu_j \varepsilon_j \in \mathfrak{U}_0$ $(t_1 \neq 0)$. Then

$$[z, x] = t_1 \sum_{i=1}^n (l_i - x_t\lambda_i) e_i + t_1 \sum_{j=1}^n (x_t\mu_j - m_j)\varepsilon_j + t_1 \sum_{i=1}^n (\lambda_i m_i - l_i \mu_i) c = 0$$

$$\iff l_i = x_t\lambda_i, m_j = x_t\mu_j \iff x \in \mathfrak{g}^+ \cap \mathfrak{g}^-.$$

By lemma 1.12, we have $\mathfrak{g}^{f_z} = \mathfrak{g}^+ \cap \mathfrak{g}^-$. So $\{\mathfrak{g}^+, \mathfrak{g}^-, f_z\}$ is a dipolarization in $\mathfrak{g}$.

Remark. The elements of $\mathfrak{U}_0$ are semisimple, the elements of $\mathfrak{U}_1$ and $\mathfrak{U}_2$ are nilpotent. So there exist nilpotent characteristic elements corresponding to some symmetric or nonsymmetric dipolarizations in extended Heisenberg algebras.

Lemma 2.1. *Let* $\mathfrak{h}_{1k}(\mu) = g_{1k} \oplus g_2 \oplus \mathbb{F}c \oplus \mathbb{F}(t + \mu e_k)$, $\mathfrak{h}_{2k}(\mu) = g_1 \oplus g_{2k} \oplus \mathbb{F}c \oplus \mathbb{F}(t + \mu\varepsilon_k)$, $\mathfrak{h}_1 = g_0 \oplus \mathbb{F}c \oplus g_1$, $\mathfrak{h}_2 = g_0 \oplus \mathbb{F}c \oplus g_2$, $\mathfrak{h}_3 = g_0 \oplus g_{1S} \oplus \overline{g}_{2S} \oplus \mathbb{F}c$, $\mathfrak{h}_4 = g_0 \oplus \overline{g}_{1S} \oplus g_{2S} \oplus \mathbb{F}c$ *and* $\mathfrak{h}_5 = g_1 \oplus g_2 \oplus \mathbb{F}c$. *Then we have* $\mathfrak{h}_{1k}(\mu) \simeq \mathfrak{h}_{1k}(\mu')$, $\mathfrak{h}_{2k}(\mu) \simeq \mathfrak{h}_{2k}(\mu')$, $\mathfrak{h}_1 \simeq \mathfrak{h}_2$, $\mathfrak{h}_3 \simeq \mathfrak{h}_4$ *but* $\mathfrak{h}_{1k}(\mu) \not\simeq \mathfrak{h}_5$, $\mathfrak{h}_{2k}(\mu) \not\simeq \mathfrak{h}_5$ *(as Lie algebras).*

Proof. Let ϕ be a linear mapping from $\mathfrak{h}_{2k}(0)$ to $\mathfrak{h}_{2k}(\mu)$ $(\mu \neq 0)$ such that $\phi(t) = t + \mu\varepsilon_k$, $\phi(e_i) = e_i + \varepsilon_k - \mu c$ $(i \neq k)$, $\phi(e_k) = e_k - \mu c$, $\phi(\varepsilon_j) = \varepsilon_j$ $(j \neq k)$ and $\phi(c) = c$. Then ϕ is a isomorphism from $\mathfrak{h}_{2k}(0)$ to $\mathfrak{h}_{2k}(\mu)$ $(\mu \neq 0)$ as Lie algebras. So $\mathfrak{h}_{2k}(\mu) \simeq \mathfrak{h}_{2k}(0)$ for any $\mu \neq 0$. Similarly, one may show that $\mathfrak{h}_{1k}(\lambda) \simeq \mathfrak{h}_{1k}(\lambda')$, $\mathfrak{h}_1 \simeq \mathfrak{h}_2$, $\mathfrak{h}_3 \simeq \mathfrak{h}_4$. Since $\mathfrak{h}_5$ is nilpotent but $\mathfrak{h}_{1k}(\mu)$ and $\mathfrak{h}_{2k}(\mu)$ are not, $\mathfrak{h}_{1k}(\mu) \not\simeq \mathfrak{h}_5$, $\mathfrak{h}_{2k}(\mu) \not\simeq \mathfrak{h}_5$. □

Proposition 2.2. *The dipolarizations* (1)–(4) *in the above table are symmetric. The dipolarizations* (5) *and* (6) *in the above table are nonsymmetric.*

Proof. It is easy to get by lemma 2.1. □

3. Parakähler manifolds associated with the dipolarizations in extended Heisenberg algebra

Let $\mathfrak{g} = \mathbb{R}t \oplus \mathbb{R}c \oplus \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_n \oplus \mathbb{R}\varepsilon_1 \oplus \cdots \oplus \mathbb{R}\varepsilon_n$ be the real quadratic $(2n+2)$-dimensional extended Heisenberg algebra with the invariant scalar product B, where $B(t, c) = 1$, $B(e_i, \varepsilon_j) = \delta_{ij}$ $(i, j = 1, 2, \ldots, n)$.

By a straight calculation, we get the matrix realization of $\mathfrak{g}$ by setting

$$t = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \qquad e_1 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$e_2 = \begin{pmatrix} 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \ldots, e_n = \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$\varepsilon_1 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \varepsilon_2 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \ldots,$$

$$\varepsilon_n = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, c = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Then $\mathfrak{g}$ is formed by the following matrices:

$$\begin{pmatrix} \lambda & x_1 & x_2 & \cdots & x_n & z \\ 0 & 0 & 0 & \cdots & 0 & y_1 \\ 0 & 0 & 0 & \cdots & 0 & y_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & y_n \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix} \qquad x_i, y_j, z, \lambda \in \mathbb{R}$$

and the connected Lie group G with this Lie algebra $\mathfrak{g}$ is formed by the matrices

$$\begin{pmatrix} \lambda & a_1 & a_2 & \cdots & a_n & \mu \\ 0 & 1 & 0 & \cdots & 0 & b_1 \\ 0 & 0 & 1 & \cdots & 0 & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & b_n \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix} \qquad a_i, b_j \in \mathbb{R}, \lambda > 0.$$

Let

$$G_1 = \left\{ \begin{pmatrix} \lambda & \lambda_1(\lambda-1) & \lambda_2(\lambda-1) & \cdots & \lambda_n(\lambda-1) & \mu \\ 0 & 1 & 0 & \cdots & 0 & \mu_1(\lambda-1) \\ 0 & 0 & 1 & \cdots & 0 & \mu_2(\lambda-1) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & \mu_n(\lambda-1) \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix} \right\}$$

where $\lambda > 0$, $\mu \in \mathbb{R}$;

$$G_{1k} = \left\{ \begin{pmatrix} 1 & a_1 & \cdots & a_n & \mu \\ 0 & 1 & \cdots & 0 & b_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & b_{k-1} \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & b_{k+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & b_n \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \right\}$$

$$G_{2k} = \left\{ \begin{pmatrix} 1 & a_1 & \cdots & a_{k-1} & 0 & a_{k+1} & \cdots & a_n & \mu \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & b_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & b_n \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \right\}$$

where $a_i, b_j, \mu \in \mathbb{R}$, $1 \leqslant k \leqslant n$.

Clearly G_1, G_{1k}, G_{2k} $(1 \leqslant k \leqslant n)$ are closed subgroups of G. It is easy to check that Lie $G_1 = \mathfrak{g}_0 \oplus \mathbb{R}c$, Lie $G_{1k} = \mathfrak{g}_0 \oplus \mathfrak{g}_{2k} \oplus \mathbb{R}c$ and Lie $G_{2k} = \mathfrak{g}_{1k} \oplus \mathfrak{g}_2 \oplus \mathbb{R}c$ $(1 \leqslant k \leqslant n)$.

Proposition 3.1. *Let G, G_1, G_{1k}, G_{2k} $(1 \leqslant k \leqslant n)$ and $\mathfrak{g}$ be as above. Then G/G_1, G/G_{1k}, G/G_{2k} have the structure of a parakähler coset space.*

Proof. We prove the assertion only for G/G_1. One can prove the other cases similarly. Let $\{\mathfrak{g}^{\pm}, f\}$ be the dipolarization (1) in $\mathfrak{g}$ as in table 1. That is

$$\mathfrak{g}^+ = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathbb{R}c \qquad \mathfrak{g}^- = \mathfrak{g}_0 \oplus \mathfrak{g}_2 \oplus \mathbb{R}c$$

and $f(e_m) = \mu_m f(c)$, $f(\varepsilon_m) = \lambda_m f(c)$ $(1 \leqslant m \leqslant n)$, $f(c) \neq 0$. Let $\rho(x, y) = f([x, y])$ for any $x, y \in \mathfrak{g}$. Then $\{\mathfrak{g}^{\pm}, \rho\}$ is a weak dipolarization in $\mathfrak{g}$ and $\mathfrak{g}^+ \cap \mathfrak{g}^- = \mathfrak{g}_0 \oplus \mathbb{R}c = $ Lie G_1.

Let

$$h = \begin{pmatrix} \lambda & \lambda_1(\lambda-1) & \lambda_2(\lambda-1) & \cdots & \lambda_n(\lambda-1) & \mu \\ 0 & 1 & 0 & \cdots & 0 & \mu_1(\lambda-1) \\ 0 & 0 & 1 & \cdots & 0 & \mu_2(\lambda-1) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & \mu_n(\lambda-1) \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix} \in G_1.$$

By a straight calculation, we have

$$\mathrm{Ad}(h)e_m = \lambda e_m + \mu_m(1-\lambda)c \qquad \mathrm{Ad}(h)\varepsilon_m = \lambda^{-1}\varepsilon_m + (1-\lambda^{-1})\lambda_m c$$

$$\mathrm{Ad}(h)\left(t + \sum_{i=1}^n \lambda_i e_i + \sum_{j=1}^n \mu_j \varepsilon_j\right) = \left(t + \sum_{i=1}^n \lambda_i e_i + \sum_{j=1}^n \mu_j \varepsilon_j\right) + \left(\sum_{i=1}^n \lambda_i \mu_i\right) c$$

$$\mathrm{Ad}(h)c = c.$$

Hence $\mathrm{Ad}(h)(\mathfrak{g}^{\pm}) \subseteq \mathfrak{g}^{\pm}$ for any $h \in G_1$.

On the other hand, for any

$$x = x_t\left(t + \sum_{i=1}^n \lambda_i e_i + \sum_{j=1}^n \mu_j \varepsilon_j\right) + x_c c + \sum_{i=1}^n x_{1i} e_i + \sum_{j=1}^n x_{2j}\varepsilon_j \in \mathfrak{g}$$

$$y = y_t\left(t + \sum_{i=1}^n \lambda_i e_i + \sum_{j=1}^n \mu_j \varepsilon_j\right) + y_c c + \sum_{i=1}^n y_{1i} e_i + \sum_{j=1}^n y_{2j}\varepsilon_j \in \mathfrak{g}$$

we have

$$\begin{aligned}\mathrm{Ad}(h)x = {} & x_t\left(t + \sum_{i=1}^n \lambda_i e_i + \sum_{j=1}^n \mu_j \varepsilon_j\right) + \lambda \sum_{i=1}^n x_{1i} e_i + \lambda^{-1}\sum_{j=1}^n x_{2j}\varepsilon_j \\ & + \left(x_c + x_t \sum_{i=1}^n \lambda_i\mu_i + (1-\lambda)\sum_{i=1}^n \mu_i x_{1i} + (1-\lambda^{-1})\sum_{j=1}^n \lambda_j x_{2j}\right) c \\ \mathrm{Ad}(h)y = {} & y_t\left(t + \sum_{i=1}^n \lambda_i e_i + \sum_{j=1}^n \mu_j \varepsilon_j\right) + \lambda \sum_{i=1}^n y_{1i} e_i + \lambda^{-1}\sum_{j=1}^n y_{2j}\varepsilon_j \\ & + \left(y_c + y_t \sum_{i=1}^n \lambda_i\mu_i + (1-\lambda)\sum_{i=1}^n \mu_i y_{1i} + (1-\lambda^{-1})\sum_{j=1}^n \lambda_j y_{2j}\right) c.\end{aligned}$$

So

$$\begin{aligned}[x, y] = {} & x_t\left(\sum_{i=1}^n y_{1i}e_i - \left(\sum_{i=1}^n \mu_i y_{1i}\right) c\right) - y_t\left(\sum_{i=1}^n x_{1i}e_i - \left(\sum_{i=1}^n \mu_i x_{1i}\right) c\right) \\ & + x_t\left(-\sum_{i=1}^n y_{2i}\varepsilon_i + \left(\sum_{i=1}^n \lambda_i y_{2i}\right) c\right) + y_t\left(\sum_{i=1}^n x_{2i}\varepsilon_i - \left(\sum_{i=1}^n \lambda_i x_{2i}\right) c\right) \\ & + \sum_{i=1}^n (x_{1i}y_{2i} - x_{2i}y_{1i})c\end{aligned}$$

$$[\mathrm{Ad}(h)x, \mathrm{Ad}(h)y] = \lambda x_t \left(\sum_{i=1}^{n} y_{1i} e_i - \left(\sum_{i=1}^{n} \mu_i y_{1i} \right) c \right) - \lambda y_t \left(\sum_{i=1}^{n} x_{1i} e_i - \left(\sum_{i=1}^{n} \mu_i x_{1i} \right) c \right)$$
$$+ \lambda^{-1} x_t \left(-\sum_{i=1}^{n} y_{2i} \varepsilon_i + \left(\sum_{i=1}^{n} \lambda_i y_{2i} \right) c \right) + \lambda^{-1} y_t \left(\sum_{i=1}^{n} x_{2i} \varepsilon_i - \left(\sum_{i=1}^{n} \lambda_i x_{2i} \right) c \right)$$
$$+ \sum_{i=1}^{n} (x_{1i} y_{2i} - x_{2i} y_{1i}) c.$$

Thus

$$\rho(\mathrm{Ad}(h)x, \mathrm{Ad}(h)y) = f([\mathrm{Ad}(h)x, \mathrm{Ad}(h)y]) = \sum_{i=1}^{n} (x_{1i} y_{2i} - x_{2i} y_{1i}) f(c)$$

$$\rho(x, y) = f([x, y]) = \sum_{i=1}^{n} (x_{1i} y_{2i} - x_{2i} y_{1i}) f(c).$$

So $\rho(\mathrm{Ad}(h)x, \mathrm{Ad}(h)y) = \rho(x, y)$ for any $h \in G_1$. By theorem 1.5, G/G_1 has the structure of a parakähler coset space. □

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